

# On the Ratio of Revenue to Welfare in Single-Parameter Mechanism Design

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## Abstract

What fraction of the potential social surplus in an environment can be extracted by a revenue-maximizing monopolist? We investigate this problem in Bayesian single-parameter environments with independent private values. The precise answer to the question obviously depends on the particulars of the environment: the feasibility constraint and the distributions from which the bidders' private values are sampled. Rather than solving the problem in particular special cases, our work aims to provide universal lower bounds on the revenue-to-welfare ratio that hold under the most general hypotheses that allow for non-trivial such bounds.

Our results can be summarized as follows. For general feasibility constraints, the revenue-to-welfare ratio is at least a constant times the inverse-square-root of the number of agents, and this is tight up to constant factors. For downward-closed feasibility constraints, the revenue-to-welfare ratio is bounded below by a constant. Both results require the bidders' distributions to satisfy hypotheses somewhat stronger than regularity; we show that the latter result cannot avoid this requirement.

## 1 Introduction

When a firm offers a new service with the potential to bring utility to a set of users, it is intuitive that the firm should be able to extract a significant fraction of that utility as profit. Is this intuition justified by theory? This fundamental question about the relation between revenue-maximizing and welfare-maximizing mechanisms is the focus of our paper.

The answer to our question depends, among other things, upon which sets of users may potentially be served. An exemplary case in which the seller's revenue is only a small fraction of the social surplus is a *public project*, in which the only two alternatives are to serve everyone or to serve no one. As we shall see in Section 3, for a public project with  $n$  agents having i.i.d. values uniformly sampled from  $[0, 1]$ , the optimal mechanism provides the seller with revenue  $\Theta(\sqrt{n})$ , whereas the expected social surplus generated by serving all agents is  $n/2$ .

There is a clear economic intuition as to why the seller's revenue is so limited in the public project setting: there is no way to deny service to one agent while serving another, so an agent's bid is unlikely to influence her own allocation. Accordingly, it is not possible to charge agents more than a small fraction of their reported value without creating an incentive for under-reporting. Pursuing this intuition further, one would expect the seller to be able to extract a much larger fraction of the potential social surplus in

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*downward-closed* environments, when the decision to deny service to an agent may be made on an individual basis.

The foregoing discussion inspires some natural questions about the relation between revenue-maximizing and welfare-maximizing mechanisms, that refine the guiding question presented at the start of the paper. Can the revenue of the optimal mechanism ever be less than  $c/\sqrt{n}$  times the expected welfare of the efficient allocation, where  $n$  is the number of agents and  $c$  is a universal constant? Under what conditions does this revenue-to-welfare ratio improve to a constant? Our goal in this paper is to answer these questions for Bayesian single-parameter environments.

A moment's thought reveals that one must place some restriction on the distributions from which the agents' values are sampled, to avoid trivialities. For example, consider a monopolist selling a single item to an agent whose value is sampled from the *equal-revenue distribution*, with cumulative distribution function satisfying  $F(x) = 1 - 1/x$  for all  $x \geq 1$ . As is well known, the seller cannot extract more than one unit of revenue, despite the fact that allocating the item yields infinite expected welfare in this case. Thus, even in the extremely simple setting a single-item auction with one agent, the seller is not guaranteed any constant fraction of the social surplus unless we make further assumptions about the distributions of agents' values.

A theme running through many of our results is that the foregoing type of distribution — one that prevents the seller in a single-item auction from extracting a constant fraction of the buyer's expected value — is essentially the only type of distribution that must be excluded in order to obtain strong lower bounds on the revenue-to-welfare ratio under arbitrary feasibility constraints. To make this more precise, for a non-negative real-valued random variable  $X$  with cumulative distribution function  $F(x)$ , let  $\rho(X)$  denote the seller's optimal revenue when selling an item to a single agent with private value  $X$ :

$$\rho(X) = \sup_{p \geq 0} \{p \cdot (1 - F(p))\}.$$

We now define the following two properties of a distribution.

**Definition 1.1.** For any number  $c > 0$ , we say a random variable  $X$  is *c-bounded* if it satisfies  $c \cdot \rho(X) \geq \mathbb{E}[X]$ , and it is *strongly c-bounded* if  $\Pr(c \cdot \rho(X) \geq X) = 1$ .

In other words, a buyer's value distribution is *c-bounded* if her expected value is at most  $c$  times the revenue that a seller can earn when selling one item to her, and it is *strongly c-bounded* if her value is *never* more than  $c$  times the seller's optimal revenue. Having made these definitions, we can state our main results. All of them pertain to Bayesian single-parameter environments in which  $n$  agents have independent private values and the feasibility constraint is specified by a set system  $\mathcal{F} \subseteq 2^{[n]}$  denoting the sets of agents that may be simultaneously served.

**Theorem 1.2.** *If  $\mathcal{F}$  is arbitrary, and all agents have strongly c-bounded distributions, then the revenue of the optimal mechanism is at least  $1/(96c\sqrt{n})$  times the expected welfare of the efficient allocation. For public project mechanisms, the same conclusion holds under the weaker hypothesis that the distributions are c-bounded.*

The following theorem refers to *hyper-regular distributions*, a mild specialization of regular distributions whose definition we defer to Section 2. All hyper-regular distributions are regular, and while the converse is not true, it is the case that most of the commonly cited examples of regular distribution — including monotone hazard rate (MHR) distributions and Pareto distributions — are hyper-regular. See the paragraph following Definition 2.1 for further discussion of this point.

**Theorem 1.3.** *If  $\mathcal{F}$  is downward-closed, and all agents have  $c$ -bounded hyper-regular distributions, then the revenue of the optimal mechanism is at least  $1/c$  times the expected welfare of the efficient allocation.*

We further show that the assumption of hyper-regularity is unavoidable in Theorem 1.3, even when dealing with single-item auctions. We give an explicit example of a regular (but not hyper-regular) distribution  $F$  such that as  $n \rightarrow \infty$ , the ratio of the optimal revenue to the maximum bid tends to zero in a single-item auction with  $n$  i.i.d. bidders sampling values from  $F$ .

To derive our results, we use a mix of techniques from economics and probability theory. Not surprisingly, we rely heavily on Myerson’s Lemma that the expected revenue of a mechanism equals its expected virtual surplus. We then face the task of proving lower bounds on the expected virtual surplus of the optimal mechanism. It turns out that this task is closely tied to proving *anti-concentration* inequalities for sums of independent random variables, i.e. inequalities asserting that the sum is unlikely to be too *close* to its expected value. We derive an anti-concentration inequality suited to our application by generalizing Erdős’s proof of the Littlewood-Offord Theorem Erdős (1945); Littlewood and Offord (1943). This inequality constitutes the main technical ingredient underlying Theorem 1.2. To obtain Theorem 1.3 we generalize a different tool from probability theory, namely Chebyshev’s Integral Inequality.

**Related work.** Many prior papers address relationships between revenue-maximizing and welfare-maximizing mechanisms in Bayesian settings. All of these papers are thematically related to our work, and some of them contain theorems that directly imply bounds on the revenue-to-welfare ratio for special cases of the settings considered here, though usually as a side effect of attacking other questions. For example, the famous work of Bulow and Klemperer (1996) shows that the revenue of the Vickrey single-item auction with  $n + 1$  i.i.d. bidders exceeds that of the optimal single-item auction with  $n$  i.i.d. bidders drawn from the same distribution, provided the distribution is regular. (Note the contrast with our work: theirs relates the *revenue* of a VCG auction to that of an optimal auction, whereas our work relates the *efficiency* of a VCG auction to the *revenue* of an optimal auction.) Drawing inspiration from Bulow and Klemperer while significantly expanding upon their techniques, Dhangwatnotai et al. (2010) designed *single sample mechanisms* and proved — under various hypotheses on the feasibility constraints and the distributions — that their mechanism’s revenue approximates that of the optimal mechanism. All of the environments considered in their paper have downward-closed feasibility constraints, unlike our paper that also addresses general feasibility constraints. Of particular relevance to our work is Theorem 3.10 of (Dhangwatnotai et al., 2010), which directly bounds the revenue-to-welfare ratio of the “VCG with lazy reserves” (VCG-L) mechanism in downward-closed environments with MHR distributions. Our Theorem 1.3 can be seen as a generalization of their Theorem 3.10 from MHR distributions to hyper-regular distributions.

Other extensions of the Bulow-Klemperer Theorem in recent years have contributed to the literature on relations between revenue-maximizing and welfare-maximizing auctions. For example, Hartline and Roughgarden (2009) consider duplicating *each* bidder, and they bound the ratio between the revenue of the VCG mechanism in the “duplicated environment” and that of the optimal mechanism in the original environment; this technique is then used to imply that simple mechanisms that modify VCG by adding reserve prices can approximate the revenue of the optimal mechanism. Extending Bulow-Klemperer in a different direction, Aggarwal et al. (2009) show that adding  $O(\log n)$  additional bidders to Myerson’s mechanism (in an i.i.d. m.h.r. single-item environment) is necessary and sufficient to achieve an expected welfare guarantee that matches that of the VCG mechanism with the original  $n$  bidders.

Other papers contributing to the literature on relationships between welfare-maximizing and revenue-maximizing mechanisms are (Daskalakis and Pierrakos, 2011), which presents auctions that simultaneously achieve good revenue and efficiency for single-item environments, and (Abhishek and Hajek, 2010), which considers the efficiency loss in revenue-maximizing mechanisms.

Our paper is not the first to use the Littlewood-Offord Theorem and its generalizations to bound the revenue of mechanisms. A different generalization of Littlewood-Offord was applied by Karlin et al. (2013) to the analysis of prior-free mechanisms.

## 2 Preliminaries

**Single-parameter Bayesian mechanism design.** In a standard single-parameter Bayesian mechanism design setting, there are  $n$  bidders or agents, each with a private value  $v_i, i = 1, \dots, n$ , denoting the value of agent  $i$  for receiving service. We will denote the cumulative distribution function of  $v_i$  by  $F_i$ , and when we assume that  $v_i$  has a density function we will denote the density function by  $f_i$ .

A general feasibility environment is specified by a set  $\mathcal{F} \subseteq 2^{[n]}$  denoting the feasible sets of bidders that can be simultaneously served. We call  $\mathcal{F}$  the *feasibility constraint* of the environment. We say  $\mathcal{F}$  is *downward-closed* if every subset of a feasible set is feasible.

A mechanism is a pair  $(A, p)$  consisting of an *allocation function*  $A : \mathbb{R}^n \rightarrow \{0, 1\}^n$  and a *payment function*  $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Both functions may possibly be randomized. The input to both functions is a vector of bids. The function  $A$  determines the set of agents who will be served; thus we require that  $\{i : A_i(b) = 1\}$  belongs to  $\mathcal{F}$  for every possible bid vector  $b$ . The payment function  $p$  determines how much each agent will pay. Agents are risk-neutral and have quasi-linear utility: an agent with value  $v_i$  who is served with probability  $\pi_i$  and pays  $p_i$  has utility  $\pi_i v_i - p_i$ .

The *expected revenue* (or simply *revenue*) of a mechanism is  $\mathbb{E}[\sum_{i=1}^n p_i(b)]$  where  $b$  is the random bid vector in some equilibrium of the mechanism. Its *expected welfare* (or simply *welfare*) is  $\mathbb{E}[\sum_i A_i(b) v_i]$ , the expected sum of values of the agents served. In both cases, the expectation is over the randomness in the agents' private values, as well as the randomness (if any) in their choice of bids and in the mechanism's choice of allocations and payments. All mechanisms in this paper are assumed to be *ex post individually rational*, meaning that agents are never charged an amount exceeding their bid.

**Probability distributions.** When  $X$  is a random variable, we denote by  $X^+ = \max\{0, X\}$  the “positive part” of  $X$ , and by  $X^- = \min\{0, X\}$  the “negative part” of  $X$ .

The hazard rate of a distribution is defined as  $h(x) = \frac{f(x)}{1-F(x)}$ , and a monotone hazard rate (MHR) distribution is one whose hazard rate is non-decreasing. The virtual valuation function corresponding to distribution  $F$  is  $\phi(x) = x - \frac{1}{h(x)}$ . Distributions with non-decreasing virtual valuation function are called regular distributions. In the sequel, we will use the following strengthening of the regularity property.

**Definition 2.1** (Hyper-regular Distribution). A hyper-regular distribution is a regular distribution with non-decreasing  $\frac{\phi(x)}{x}$ .

Most of the common examples of regular distributions are actually hyper-regular. For example, it is easy to see that all MHR distributions are hyper-regular. Also, Pareto distributions having cumulative density function  $F(x) = 1 - x^{-\alpha}$ , where  $\alpha > 1$  (a necessary condition for the distribution to be regular, and also for it to have finite expected value) are hyper-regular. Not all regular distributions are hyper-regular; for example, the distribution specified by  $F(x - \delta) = 1 - \frac{1}{x \ln^2 x}$ , where  $\delta \ln^2 \delta = 1$ , is not hyper-regular. We will return to this distribution at the end of Section 5.

### Myerson's lemma.

Myerson (1981) gave a connection between the expected revenue and the expected virtual surplus.

**Lemma 2.2** (Myerson’s Lemma). *In a truthful mechanism  $(A, p)$  the expected payment  $p_i$  of agent  $i$  with virtual valuation function  $\phi_i$  satisfies:*

$$\mathbb{E}[p_i(\mathbf{v})] = \mathbb{E}[\phi_i(v_i) \cdot A_i(\mathbf{v})]$$

*The equality holds even when the bids of other bidders  $v_{-i}$  are fixed.*

Thus, when virtual surplus maximization induces a monotone allocation rule, this allocation rule maximizes revenue. This criterion is always satisfied when bidders’ values are drawn from regular distributions. When the distributions are not regular, Myerson provides a workaround: an *ironed virtual valuation function*  $\bar{\phi}_i$  for each bidder, which is always monotone, such that Myerson’s Lemma continues to hold provided that the allocation rule is constant on any interval in which the bidder’s ironed virtual value is constant. Ironed virtual surplus maximization induces a monotone allocation rule, and a mechanism with this allocation rule maximizes revenue.

To maximize the welfare, we can use the well-known VCG mechanism. In this paper we also use a variation of the VCG mechanism called “VCG with lazy reserves”, or simply VCG-L (Dhangwatnotai et al., 2010), which operates as follows:

1. Run the VCG mechanism to obtain a preliminary winning set  $P$ .
2. Remove all the bidders  $i \in P$  with  $v_i < r_i$ , where  $r_i = \phi_i^{-1}(0)$  is the reserve price for the bidder  $i$ .
3. Charge each winning bidder  $i$  the larger of  $r_i$  and its VCG payment in the first step.

### 3 Warm-up: Identical uniform distributions

As a prelude to our main results, we devote this section to bounding the revenue-to-welfare ratio when the bids are i.i.d. uniform samples from  $[0,1]$ . The results in this section will be completely subsumed by subsequent theorems, but they have much simpler proofs that serve to illustrate the main ideas underlying our later results while highlighting the technical challenges that must be overcome in order to prove those more general results.

The uniform distribution on  $[0,1]$  has a very simple virtual valuation function. We have  $F(x) = x$  and  $f(x) = 1$  for all  $x \in [0, 1]$ , and so

$$\phi(x) = x - \frac{1 - F(x)}{f(x)} = x - (1 - x) = 2x - 1.$$

The following simple consequence is important for our analysis.

$$\text{If } x \text{ is uniformly distributed in } [0, 1] \text{ then } \phi(x) \text{ is uniformly distributed in } [-1, 1]. \quad (*)$$

Let us first use these observations to derive an asymptotic expression for the revenue-to-welfare ratio for a public project with  $n$  i.i.d. uniform  $[0,1]$  bids. The allocation that provides service to all bidders also maximizes welfare, so the expected welfare of the efficient allocation is simply:  $\mathbb{E}[x_1 + \dots + x_n] = \frac{n}{2}$ . The virtual surplus is maximized by serving everyone if  $\phi_1(x_1) + \dots + \phi_n(x_n) \geq 0$ , and otherwise by serving no one. Therefore, the optimal mechanism’s revenue is  $\mathbb{E}[(\phi_1(x_1) + \dots + \phi_n(x_n))^+]$ . An asymptotic formula for this expression can readily be computed using the Central Limit Theorem. The random variables  $\phi_i(x_i)$

are i.i.d. uniform samples from  $[-1, 1]$ , so they have mean zero and variance  $\sigma^2 = \frac{1}{3}$ . Consequently the random variable  $n^{-1/2} \sum_{i=1}^n \phi_i(x_i)$  converges in distribution to  $\mathcal{N}(0, 1/3)$ , and thus

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{\sqrt{n}} \mathbb{E} [(\phi_1(x_1) + \dots + \phi_n(x_n))^+] \right\} = \frac{1}{\sqrt{6\pi}} \int_0^\infty t e^{-t^2/2} dt = \frac{1}{\sqrt{6\pi}}.$$

Recalling that the expectation of the maximum welfare in this case is  $n/2$ , we see that the revenue-to-welfare ratio is asymptotic to  $\sqrt{\frac{2}{3\pi n}}$ , and in particular it is  $\Theta(n^{-1/2})$ .

Let us now generalize to arbitrary feasibility constraints. Intuitively, it seems that the revenue-to-welfare ratio should be minimized by the public project environment, for the reasons articulated in the introduction. The following proposition confirms that this intuition is valid, at least up to a constant factor.<sup>1</sup>

**Proposition 3.1.** *For a Bayesian single-parameter environment with  $n$  i.i.d. bidders having uniform  $[0, 1]$  values, and a general feasibility constraint  $\mathcal{F}$ , the revenue-to-welfare ratio is always at least  $\Omega(n^{-1/2})$ .*

*Proof.* Among all feasible sets, let  $S^*$  be one with maximum cardinality,  $k = |S^*|$ . We will show that the revenue-to-welfare ratio is  $\Omega(k^{-1/2})$ , from which the proposition follows *a fortiori*.

As the bidders' values are never greater than 1, the welfare of the efficient allocation is never greater than  $k$ . Consider a mechanism  $\mathcal{M}$  which maximizes revenue subject to the constraint that the set of agents served is always either  $\emptyset$  or  $S^*$ . This is simply an optimal mechanism for a public project with agent set  $S^*$ , so we have already calculated that its revenue is  $\Theta(k^{1/2})$ . The revenue of the optimal mechanism is at least as great as that of  $\mathcal{M}$ , hence the revenue-to-welfare ratio is  $\Omega(k^{1/2}/k) = \Omega(k^{-1/2})$ , as claimed.  $\square$

When the feasibility constraint is downward closed, and bids are i.i.d. uniform in  $[0, 1]$ , an even easier argument establishes that the revenue-to-welfare ratio is  $\Omega(1)$ .

**Proposition 3.2.** *For a Bayesian single-parameter environment with  $n$  i.i.d. bidders having uniform  $[0, 1]$  values, and a downward-closed feasibility constraint  $\mathcal{F}$ , the revenue-to-welfare ratio is always at least  $\frac{1}{4}$ .*

*Proof.* As before, define  $S^* \in \mathcal{F}$  to be a feasible set of maximum cardinality, and let  $k = |S^*|$ . The welfare of the efficient allocation is never greater than  $k$ , and we will prove that the revenue of the optimal mechanism is at least  $k/4$ .

Let  $\mathcal{M}'$  be the mechanism that maximizes revenue subject to the constraint that the set of agents served is always a subset of  $S^*$ . By Myerson's Lemma, the expected revenue of  $\mathcal{M}'$  is simply  $\sum_{i \in S^*} \mathbb{E}[\phi_i(x_i)^+]$ . Recalling that  $\phi_i(x_i)$  is uniformly distributed in  $[-1, 1]$ , we see that  $\mathbb{E}[\phi_i(x_i)^+] = \frac{1}{4}$  for each  $i$ , and the result follows.  $\square$

As we aim to extend these results to general distributions, it is worthwhile to reflect on the aspects of the proofs that were specific to the uniform distribution.

1. Our analysis of the revenue-to-welfare ratio of the public project hinged on deriving the asymptotic lower bound  $\mathbb{E}[(\phi_1(x_1) + \dots + \phi_n(x_n))^+] = \Theta(\sqrt{n})$ . We achieved this using the Central Limit Theorem. To extend this step to more general — and not necessarily identical — distributions, we require what might be called *anti-concentration* inequalities for sums of independent random variables. Versions of the Central Limit Theorem for non-identical distributions exist, but they are not general enough for our purposes. (For instance, they require upper bounds on the second moments, whereas we do not.) Instead we will generalize a different anti-concentration inequality, the Littlewood-Offord Theorem.

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<sup>1</sup>In fact, the argument given in the proof shows that the public project minimizes the revenue-to-welfare ratio up to a factor of 2. It is an interesting open question whether the revenue-to-welfare ratio is *precisely* minimized by the public project.

2. In the proofs of both propositions, we bounded the expected welfare of the efficient allocation by the cardinality of the maximum feasible set. This very simple upper-bounding technique was effective because the uniform distribution is strongly  $c$ -bounded for  $c = 2$ . (The expected welfare of any set of agents is at least half of its cardinality.) When dealing with distributions that are not strongly  $c$ -bounded, we need to develop a different technique for upper-bounding the expected welfare of the efficient allocation.

## 4 General feasibility constraints

In this section, we consider arbitrary feasibility constraints with  $n$  agents and extend the  $\Omega(n^{-1/2})$  lower bound on the revenue-to-welfare ratio (Proposition 3.1) from i.i.d. uniform bids to more general distributions. As noted at the end of Section 3, the key to proving such an extension is to derive an inequality asserting that the distribution of a sum of independent random variables cannot be too tightly concentrated around its expected value. We first derive a suitably general inequality in Subsection 4.1, and we apply this inequality in the following subsections.

### 4.1 A generalization of the Littlewood-Offord Theorem

A beautiful “anti-concentration” inequality for independent random variables was proven by Littlewood and Offord (1943) and strengthened by Erdős (1945).

**Theorem 4.1** ((Littlewood and Offord, 1943; Erdős, 1945)). *For any real numbers  $x_1, \dots, x_n \geq 1$  and any half-open interval  $I$  of length 2, the number of sums  $\sum_{i=1}^n \epsilon_i x_i$  that belong to  $I$  as the vector  $(\epsilon_1, \dots, \epsilon_n)$  ranges over  $\{\pm 1\}^n$ , is at most  $\binom{n}{\lfloor n/2 \rfloor}$ .*

In this section we present a more general anti-concentration inequality for sums of independent random variables. To state our generalization, we must first specify a few notations concerning deviations of random variables.

**Definition 4.2.** For a random variable  $X$ , we define its *median*  $m(X)$  to be any number such that  $\Pr(X < m(X)) \leq 1/2$  and  $\Pr(X > m(X)) \leq 1/2$ . We will denote the absolute deviation of  $X$  from its mean and median by

$$\begin{aligned} \text{MD}(X) &= \mathbb{E}|X - \mathbb{E}X| \\ \text{MDM}(X) &= \mathbb{E}|X - m(X)|. \end{aligned}$$

Note that if there is more than one number  $m(X)$  satisfying the definition of the median of  $X$ , then the value of  $\text{MDM}(X)$  is independent of the choice of  $m(X)$ .

The following simple relations between  $\text{MD}(X)$  and  $\text{MDM}(X)$  are proven in Appendix A

**Lemma 4.3.** *For any random variable  $X$  and any constant  $a$ ,*

$$\text{MDM}(X - a) = \text{MDM}(a - X) = \text{MDM}(X) \quad \text{and} \quad \text{MD}(X - a) = \text{MD}(a - X) = \text{MD}(X). \quad (1)$$

Furthermore,

$$\text{MDM}(X) \leq \text{MD}(X) \leq 2\text{MDM}(X). \quad (2)$$

Our first anti-concentration result is stated in the following proposition, whose proof is also deferred to Appendix A.

**Proposition 4.4.** *If  $X_1, \dots, X_n$  are independent random variables and  $\text{MDM}(X_i) \geq 1$  for all  $i$ , then  $\text{MDM}(X_1 + \dots + X_n) \geq \frac{1}{12}\sqrt{n}$ .*

We leverage the proposition to derive the following result.

**Theorem 4.5.** *Let  $Y_1, \dots, Y_n$  be any  $n$ -tuple of independent random variables, each with expectation zero. Let  $z_i = \mathbb{E}[Y_i^+]$  for  $i = 1, \dots, n$ . Then*

$$\mathbb{E}[(Y_1 + \dots + Y_n)^+] \geq \frac{z_1 + \dots + z_n}{48\sqrt{n}}. \quad (3)$$

*Proof.* Assume, without loss of generality, that  $z_1 \geq z_2 \geq \dots \geq z_n$ . Also assume that  $\max_{1 \leq k \leq n} \{z_k \sqrt{k}\} = 1$ . The latter assumption is without loss of generality because we can rescale all the random variables  $Y_1, \dots, Y_n$  by the same positive scalar without affecting the lemma's hypotheses or conclusion.

Our assumption that  $\max\{z_k \sqrt{k}\} = 1$  implies that  $z_k \leq k^{-1/2}$  for all  $k$ , hence

$$z_1 + \dots + z_n \leq \sum_{k=1}^n k^{-1/2} < 2\sqrt{n}.$$

Consequently, the value of  $(z_1 + \dots + z_n)/(12\sqrt{n})$  is bounded above by  $\frac{1}{6}$ . If we can show that the expected value of  $|Y_1 + \dots + Y_n|$  is bounded below by a constant, we are done, since the relation  $\mathbb{E}[(Y_1 + \dots + Y_n)^+] = \frac{1}{2}\mathbb{E}|Y_1 + \dots + Y_n|$  holds for the mean-zero random variable  $Y_1 + \dots + Y_n$ .

We know, from Lemma 4.3, that for all  $i$ ,  $\text{MDM}(Y_i) \geq \frac{1}{2}\mathbb{E}|Y_i| = z_i$ . Applying Proposition 4.4 to the random sum  $Y_1 + \dots + Y_k$ , it follows that the expected absolute value of that sum is at least  $\frac{1}{12}z_k \sqrt{k} = \frac{1}{12}$ .

Next we show that  $\mathbb{E}|Y_1 + \dots + Y_n| \geq \mathbb{E}|Y_1 + \dots + Y_k|$ . We have

$$\mathbb{E}[\text{sgn}(Y_1 + \dots + Y_n) \cdot (Y_1 + \dots + Y_n)] = \mathbb{E}|Y_1 + \dots + Y_n| \quad (4)$$

$$\begin{aligned} \mathbb{E}[\text{sgn}(Y_1 + \dots + Y_k) \cdot (Y_1 + \dots + Y_n)] &= \mathbb{E}|Y_1 + \dots + Y_k| \\ &\quad + \mathbb{E}[\text{sgn}(Y_1 + \dots + Y_k) \cdot (Y_{k+1} + \dots + Y_n)] \end{aligned} \quad (5)$$

$$= \mathbb{E}|Y_1 + \dots + Y_k| \quad (6)$$

where the last equality holds because  $Y_1 + \dots + Y_k$  is independent of  $Y_{k+1} + \dots + Y_n$ , and the latter has zero expected value.

The left side of (4) is greater than or equal to the left side of (3), because the inequality

$$[\text{sgn}(Y_1 + \dots + Y_n) - \text{sgn}(Y_1 + \dots + Y_k)] \cdot (Y_1 + \dots + Y_n) \geq 0$$

holds for all values of  $Y_1, \dots, Y_n$ . Indeed, whenever the quantity  $[\text{sgn}(Y_1 + \dots + Y_n) - \text{sgn}(Y_1 + \dots + Y_k)]$  is nonzero, it has the same sign as  $Y_1 + \dots + Y_n$ . Combining previous steps, we obtain

$$\mathbb{E}|Y_1 + \dots + Y_n| \geq \mathbb{E}|Y_1 + \dots + Y_k| \geq \frac{1}{12} \geq \frac{z_1 + \dots + z_n}{24\sqrt{n}},$$

and the theorem follows since  $\mathbb{E}[(Y_1 + \dots + Y_n)^+] = \frac{1}{2}\mathbb{E}|Y_1 + \dots + Y_n|$ .  $\square$



**Theorem 4.6.** *Let  $Y_1, \dots, Y_n$  be any  $n$ -tuple of independent random variables with positive expectations  $y_1, \dots, y_n$ . Let  $z_i = \mathbb{E}[Y_i^+]$  for  $i = 1, \dots, n$ . Then*

$$\mathbb{E}[(Y_1 + \dots + Y_n)^+] \geq \frac{z_1 + \dots + z_n}{96\sqrt{n}}. \quad (7)$$

*Proof.* We write  $Y_i = Y'_i + y_i$  and  $z_i = z'_i + y_i$ , for each  $i$ . Then the expectation of  $Y'_i$  is zero, and according to Theorem 4.5, we know that

$$E[(Y_1 + \dots + Y_n - \sum_i y_i)^+] \geq \frac{z_1 + \dots + z_n - \sum_i y_i}{48\sqrt{n}}$$

Note  $\sum_i y_i$  is positive, so the inequality above gives a lower bound on  $E[(Y_1 + \dots + Y_n)^+]$ .

In addition, we know

$$E[(Y_1 + \dots + Y_n)^+] \geq \sum_i y_i,$$

as otherwise the expectation of  $Y_1 + \dots + Y_n$  would be less than  $\sum_i y_i$ , a contradiction.

Thus,

$$\begin{aligned} E[(Y_1 + \dots + Y_n)^+] &\geq \max \left\{ \frac{z_1 + \dots + z_n - \sum_i y_i}{48\sqrt{n}}, \sum_i y_i \right\} \\ &\geq \frac{1}{2} \left( \frac{z_1 + \dots + z_n - \sum_i y_i}{48\sqrt{n}} + \sum_i y_i \right) \geq \frac{z_1 + \dots + z_n}{96\sqrt{n}} \end{aligned}$$

□

## 4.2 Public projects

In this section we analyze the revenue-to-welfare ratio for a public project with  $c$ -bounded distributions, as a step toward analyzing environments with general feasibility constraints.

**Proposition 4.7.** *In a public project environment whose  $n$  agents have independent  $c$ -bounded distributions, the revenue of the optimal mechanism is at least  $1/(96c\sqrt{n})$  times the expected welfare of the efficient allocation.*

*Proof.* For each agent  $i$ , recall that  $\bar{\phi}_i$  denotes the agent's ironed virtual valuation function and that  $\rho(v_i) = \mathbb{E}[\bar{\phi}_i(v_i)^+]$  denotes the maximum revenue that a monopolist can obtain by selling a single item to agent  $i$ ; denote this number by  $\rho_i$  henceforth in the proof. Our assumption that the value distribution  $F_i$  is  $c$ -bounded implies that  $\mathbb{E}v_i \leq c\rho_i$ , so the expected welfare of the efficient allocation is bounded by  $c(\rho_1 + \dots + \rho_n)$ .

Applying Theorem 4.6 to the random variables  $Y_i = \bar{\phi}_i(v_i)$  we conclude that

$$\mathbb{E}[(Y_1 + \dots + Y_n)^+] \geq \frac{\rho_1 + \dots + \rho_n}{96\sqrt{n}}. \quad (8)$$

This completes the proof, since the left side is the optimal mechanism's expected revenue. □

### 4.3 Strongly $c$ -bounded distributions

In this section we prove the first part of Theorem 1.2, which deals with arbitrary feasibility constraints and strongly  $c$ -bounded distributions. The second part of the theorem, which deals with public projects, was already proven in Proposition 4.7 above.

**Proposition 4.8.** *If  $\mathcal{F}$  is arbitrary, and all agents have strongly  $c$ -bounded distributions, then the revenue of the optimal mechanism is at least  $1/(96c\sqrt{n})$  times the expected welfare of the efficient allocation.*

*Proof.* As in the preceding proof, let  $Y_i = \bar{\phi}_i(v_i)$  and  $\rho_i = \mathbb{E}[Y_i^+]$  for each agent  $i$ . Our assumption that the value distribution  $F_i$  is strongly  $c$ -bounded implies that  $v_i$  is never greater than  $c\rho_i$ . For any set of agents  $S$ , let  $\rho(S) = \sum_{i \in S} \rho_i$  and define  $S^*$  to be a feasible set that maximizes  $\rho$ . Let  $k = |S^*|$ . We will show that the revenue-to-welfare ratio is at least  $1/(96c\sqrt{k})$ , from which the proposition follows *a fortiori*.

As bidder  $i$ 's value never exceeds  $c\rho_i$ , the value of any allocation  $S \in \mathcal{F}$  never exceeds  $c\rho(S)$ , which is in turn bounded above by  $c\rho(S^*)$ . Hence  $c\rho(S^*)$  is an upper bound on the expected welfare of the efficient allocation.

Consider a mechanism  $\mathcal{M}$  which maximizes revenue subject to the constraint that the set of agents served is always either  $\emptyset$  or  $S^*$ . This is simply an optimal mechanism for a public project with agent set  $S^*$ , so its expected revenue is  $\mathbb{E} \left[ \left( \sum_{i \in S^*} Y_i \right)^+ \right]$ . Theorem 4.6 guarantees that

$$\mathbb{E} \left[ \left( \sum_{i \in S^*} Y_i \right)^+ \right] \geq \frac{\rho(S^*)}{96c\sqrt{k}},$$

and the proof is complete. □

## 5 Downward-closed environments

In this section we consider the revenue-to-welfare ratio of environments with downward-closed feasibility constraints. Our main result shows that the optimal mechanism's revenue is a  $\Omega(1/c)$  fraction of the expected welfare of the efficient allocation, when the distributions are  $c$ -bounded and hyper-regular. Recall that a hyper-regular distribution is one such that  $\phi(x)/x$  is a non-decreasing function of  $x$ , where  $\phi$  denotes the virtual value function.

### 5.1 An inequality for monotonic functions of a random variable

As in Section 4, a probabilistic inequality lies at the heart of our main result. In this case, the inequality in question is a generalization of Chebyshev's Integral Inequality (Fink and Jodeit, 1984), which asserts that for two monotonically non-decreasing function  $f, g$  on an interval  $(a, b)$ ,

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \geq \left[ \frac{1}{b-a} \int_a^b f(x) dx \right] \left[ \frac{1}{b-a} \int_a^b g(x) dx \right].$$

Our generalization is the following lemma.

**Lemma 5.1.** *Suppose  $f, g, h$  are three functions of a real number  $x$ , such that  $f, g$  are both monotonically non-decreasing, and  $h(x) \geq 0$  for all  $x$ . Then for any random variable  $X$  such that  $\mathbb{E}[h(X)] > 0$ , we have:*

$$\frac{\mathbb{E}[f(X)g(X)h(X)]}{\mathbb{E}[g(X)h(X)]} \geq \frac{\mathbb{E}[f(X)h(X)]}{\mathbb{E}[h(X)]}.$$

The version of Chebyshev's Integral Inequality stated above is obtained by setting  $h(x) = 1$  and taking  $X$  to be uniformly distributed in  $(a, b)$ . We now present the proof of the lemma.

*Proof.* Let  $c = \frac{\mathbb{E}[f(X)h(X)]}{\mathbb{E}[h(X)]}$ . The inequality  $\frac{\mathbb{E}[f(X)g(X)h(X)]}{\mathbb{E}[g(X)h(X)]} \geq c$  is equivalent to

$$\mathbb{E}[(f(X) - c)g(X)h(X)] \geq 0,$$

which we now prove. Since  $f(x)$  is non-decreasing, there is a value  $x_0$  such that  $f(x) \leq c$  for  $x < x_0$  and  $f(x) \geq c$  for  $x > x_0$ . Since  $g(x)$  is also non-decreasing, the inequality  $(f(x) - c) \cdot (g(x) - g(x_0)) \geq 0$  holds for all  $x$ . Rewrite this inequality as  $(f(x) - c) \cdot g(x) \geq (f(x) - c) \cdot g(x_0)$  and use it to deduce:

$$\mathbb{E}[(f(X) - c)g(X)h(X)] \geq \mathbb{E}[(f(X) - c)g(x_0)h(X)] \geq g(x_0) \cdot \{\mathbb{E}[f(X)h(X)] - c\mathbb{E}[h(X)]\} = 0,$$

which proves the lemma.  $\square$

## 5.2 A revenue-to-welfare bound for hyper-regular distributions

**Theorem 5.2.** *If  $\mathcal{F}$  is downward-closed, and all agents have  $c$ -bounded hyper-regular distributions, then the revenue of the optimal mechanism is at least  $1/c$  times the expected welfare of the efficient allocation.*

*Proof.* Fix any allocation  $S$ , and let  $\mathbf{1}_{S=\text{opt}}$  denote the indicator random variable of the event that  $S$  is the welfare-maximizing allocation. For any bidder  $i$  let

$$g_i(x) = \mathbb{E}[\mathbf{1}_{S=\text{opt}} \mid v_i = x] = \Pr(S = \text{opt} \mid v_i = x).$$

Note that for every  $i \in S$ , the function  $g_i(x)$  is non-decreasing for the simple reason that if  $i$  is a bidder in the welfare-maximizing set and her value increases, the welfare-maximizing set remains the same.

For any  $i \in S$  let us apply Lemma 5.1 to the functions  $f(x) = \phi_i(x)^+/x$ ,  $g(x) = g_i(x)$ ,  $h(x) = x$ , and the random variable  $X = v_i$ . The function  $f(x)$  is non-decreasing because bidder  $i$  has a hyper-regular distribution, and the function  $g(x)$  was proven to be non-decreasing in the first paragraph of this proof. Thus, we conclude that

$$\frac{\mathbb{E}[\phi_i(v_i)^+ g_i(v_i)]}{\mathbb{E}[v_i g_i(v_i)]} \geq \frac{\mathbb{E}[\phi_i(v_i)^+]}{\mathbb{E}[v_i]}. \quad (9)$$

The right side of the inequality is at least  $\frac{1}{c}$ , because  $v_i$  is sampled from a  $c$ -bounded distribution. To interpret the left side, recall the definition of  $g_i$ . We have:

$$\frac{\mathbb{E}[\phi_i(v_i)^+ g_i(v_i)]}{\mathbb{E}[v_i g_i(v_i)]} = \frac{\mathbb{E}[\phi_i(v_i)^+ \mathbb{E}[\mathbf{1}_{S=\text{opt}} \mid v_i]]}{\mathbb{E}[v_i \mathbb{E}[\mathbf{1}_{S=\text{opt}} \mid v_i]]} = \frac{\mathbb{E}[\phi_i(v_i)^+ \mathbf{1}_{S=\text{opt}}]}{\mathbb{E}[v_i \mathbf{1}_{S=\text{opt}}]}. \quad (10)$$

Combining (9) with (10) and recalling that the right side of (9) is at least  $\frac{1}{c}$ , we have derived:

$$\mathbb{E}[\phi_i(v_i)^+ \mathbf{1}_{S=\text{opt}}] \geq \frac{1}{c} \mathbb{E}[v_i \mathbf{1}_{S=\text{opt}}]. \quad (11)$$

Summing over all  $i \in S$  and using the notations  $\phi^+(S) = \sum_{i \in S} \phi_i(v_i)^+$ ,  $v(S) = \sum_{i \in S} v_i$ , we obtain

$$\mathbb{E}[\phi^+(S) \mathbf{1}_{S=\text{opt}}] \geq \frac{1}{c} \mathbb{E}[v(S) \mathbf{1}_{S=\text{opt}}]. \quad (12)$$

Finally, summing over all feasible sets  $S \in \mathcal{F}$ , we find that

$$\mathbb{E}[\phi^+(\text{opt})] \geq \frac{1}{c} \mathbb{E}[v(\text{opt})]. \quad (13)$$

The right side is the expected welfare of the efficient allocation. The left side is the expected revenue of the mechanism that selects the efficient allocation and then removes agents whose virtual value is negative, i.e. the VCG-L mechanism that was defined at the end of Section 2. The expected revenue of the optimal mechanism is at least as great as that of VCG-L, so our theorem is proved.  $\square$

Based on the proof, we immediately have the following corollary:

**Corollary 5.3.** *If  $\mathcal{F}$  is downward-closed, and all agents have  $c$ -bounded hyper-regular distributions, then the revenue of the VCG-L Mechanism is at least  $1/c$  times the expected welfare of the efficient allocation.*

### 5.3 Ratio for non-hyper-regular distributions

We use an example to show that even in the setting of a single item auction with  $n$  i.i.d. bidders, the revenue-to-welfare ratio for  $c$ -bounded regular distributions that are not hyper-regular may tend to zero as  $n$  grows to infinity. Defining  $\delta > 0$  by the equation  $\delta \ln^2 \delta = 1$ , our distribution has cumulative distribution function

$$F(x - \delta) = 1 - \frac{1}{x \ln^2 x}.$$

(Our choice of  $\delta$  is to ensure that  $F(x) \geq 0$  for all  $x \geq 0$ .) A random variable  $X$  with this distribution satisfies  $\mathbb{E}[X] = 1/(\ln \delta) < 1.5$  while  $\rho(X) > 0.25$ , so the distribution is  $c$ -bounded for any  $c \geq 6$ .

By computing the density

$$f(x - \delta) = (1 - F(x - \delta))' = \frac{\ln x + 2}{x^2 \ln^3 x},$$

we find that

$$\phi(x - \delta) = x - \delta - \frac{1 - F(x - \delta)}{f(x - \delta)} = x - \delta - \frac{x \ln x}{\ln x + 2} = -\delta + \frac{2x}{\ln x + 2}.$$

Note that  $\phi$  is an increasing function; the distribution is regular. Let

$$Z = F^{-1}\left(\frac{n-1}{n}\right) > \frac{n}{\ln^2(n)} - \delta.$$

If  $X_1, \dots, X_n$  are i.i.d. random variables with distribution  $F$  and  $X^* = \max\{X_1, \dots, X_n\}$  then the event  $X^* < Z$  has probability  $(\frac{n-1}{n})^n < 1/e$ . By Myerson's Lemma, the revenue of the optimal mechanism equals  $\mathbb{E}[\phi(X^*)^+]$ . An upper bound on this quantity can be derived as follows.

$$\begin{aligned} \mathbb{E}[\phi(X^*)^+] &= \mathbb{E}[\phi(X^*)^+ \mathbf{1}_{Z \leq X^*}] + \mathbb{E}[\phi(X^*)^+ \mathbf{1}_{Z > X^*}] \\ &\leq \mathbb{E}[\phi(X^*)^+ \mathbf{1}_{Z \leq X^*}] + \phi(Z)^+ \Pr(Z > X^*) \\ &= \mathbb{E}[\phi(X^*)^+ \mathbf{1}_{Z \leq X^*}] + \frac{1}{e-1} \phi(Z)^+ \Pr(Z \leq X^*) \\ &= \mathbb{E}\left[\left(\phi(X^*)^+ + \frac{1}{e-1} \phi(Z)^+\right) \mathbf{1}_{Z \leq X^*}\right] \\ &\leq \frac{e}{e-1} \mathbb{E}[\phi(X^*)^+ \mathbf{1}_{Z \leq X^*}] \\ &\leq \frac{e}{e-1} \mathbb{E}\left[\frac{2X^*}{\ln(X^* + \delta) + 2} \mathbf{1}_{Z \leq X^*}\right] && (\text{since } \phi(x) > \frac{2x}{\ln(x+\delta)+2} \text{ for all } x) \\ &\leq \frac{e}{e-1} \cdot \frac{2}{\ln(Z + \delta) + 2} \mathbb{E}[X^* \mathbf{1}_{Z \leq X^*}] \\ &\leq \frac{e}{e-1} \cdot \frac{2}{\ln(n/\ln^2(n)) + 2} \mathbb{E}[X^*] \end{aligned}$$

The revenue-to-welfare ratio  $\mathbb{E}[\phi(X^*)^+]/\mathbb{E}[X^*]$  therefore converges to zero as  $n \rightarrow \infty$ .

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## References

- V. Abhishek and B. E. Hajek. Efficiency loss in revenue optimal auctions. In *Proceedings of the 49th IEEE Conference on Decision and Control, CDC 2010*, pages 1082–1087, 2010.
- G. Aggarwal, G. Goel, and A. Mehta. Efficiency of (revenue-)optimal mechanisms. In *Proceedings 10th ACM Conference on Electronic Commerce (EC-2009)*, pages 235–242, 2009.
- J. Bulow and P. Klemperer. Auctions versus negotiations. *American Economic Review*, 86(1):180–94, March 1996.
- C. Daskalakis and G. Pierrakos. Simple, optimal and efficient auctions. In *Proceedings of the 7th International Workshop on Internet and Network Economics (WINE-2011)*, pages 109–121, Berlin, Heidelberg, 2011. Springer-Verlag. ISBN 978-3-642-25509-0.
- P. Dhangwatnotai, T. Roughgarden, and Q. Yan. Revenue maximization with a single sample. In *Proceedings of the 11th ACM Conference on Electronic Commerce (EC-2010)*, pages 129–138, New York, NY, USA, 2010. ACM. ISBN 978-1-60558-822-3.
- P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 52:898–902, 1945.
- A. M. Fink and M. Jodeit. On Chebyshev’s other inequality. In Y. L. Tong and S. S. Gupta, editors, *Inequalities in Statistics and Probability: Proceedings of the Symposium on Inequalities in Statistics and Probability, October 2730, 1982, Lincoln, Nebraska*, volume 5 of *Institute of Mathematical Statistics Lecture Notes - Monograph Series*, pages 115–120. 1984.
- J. D. Hartline and T. Roughgarden. Simple versus optimal mechanisms. In *Proceedings of the 10th ACM Conference on Electronic Commerce (EC-2009)*, pages 225–234, 2009.
- A. R. Karlin, T. Nguyen, and Y. Peres. Selling in exclusive markets: Some observations on prior-free mechanism design. In *ACM Transactions on Economics and Computation*, 2013.
- J. E. Littlewood and A. C. Offord. On the number of real roots of a random algebraic equation. iii. *Rec. Math. [Mat. Sbornik] N.S.*, 12:277–286, 1943.
- R. B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):pp. 58–73, 1981. ISSN 0364765X.
- E. Sperner. Ein satz uber untermengen einer endlichen menge. *Math. Zeitschrift*, 27:544–548, 1928.

## A Appendix: Proof of Proposition 4.4

We begin this appendix with a restatement and proof of Lemma 4.3.

**Lemma A.1.** *For any random variable  $X$  and any constant  $a$ ,*

$$\text{MDM}(X - a) = \text{MDM}(a - X) = \text{MDM}(X) \quad \text{and} \quad \text{MD}(X - a) = \text{MD}(a - X) = \text{MD}(X). \quad (14)$$

Furthermore,

$$\text{MDM}(X) \leq \text{MD}(X) \leq 2\text{MDM}(X). \quad (15)$$

*Proof.* The relations (14) are immediate from the definitions. To prove the inequalities in (15), it suffices to consider the case when  $m(X) \leq 0 = \mathbb{E}X$ , since the general case can then be derived by setting  $a = \mathbb{E}X$  and applying the relations (14). Let  $m = m(X)$ . Under the hypothesis that  $m \leq 0 = \mathbb{E}X$ , we have

$$\text{MD}(X) = \mathbb{E}|X| = 2\mathbb{E}[X^+] \leq 2\mathbb{E}[(X - m)^+] \leq 2\text{MDM}(X),$$

which establishes one of the two inequalities in (15). To prove the other one, we use the relation  $|x| = x \cdot \text{sgn}(x)$  to obtain

$$\begin{aligned} \text{MD}(X) - \text{MDM}(X) &= \mathbb{E}[X \cdot \text{sgn}(X) - (X - m) \cdot \text{sgn}(X - m)] \\ &= \mathbb{E}[X \cdot (\text{sgn}(X) - \text{sgn}(X - m))] + m \cdot \mathbb{E}[\text{sgn}(X - m)] \\ &= \mathbb{E}[X \cdot (\text{sgn}(X) - \text{sgn}(X - m))] \geq 0, \end{aligned}$$

where the last inequality follows because  $\text{sgn}(X) - \text{sgn}(X - m)$  is non-zero only when  $m \leq X \leq 0$ , in which case both  $X$  and  $\text{sgn}(X) - \text{sgn}(X - m)$  are non-positive.  $\square$

We now commence the proof of Proposition 4.4. We first need a definition and some preliminary lemmas.

**Definition A.2.** If  $\epsilon$  is a  $\{\pm 1\}$ -valued random variable we say that  $\epsilon$  is *medially coupled* to  $X$  if  $\Pr(\epsilon = 1) = \Pr(\epsilon = -1) = \frac{1}{2}$  and  $\epsilon(X - m(X))$  is always non-negative. Note that this is equivalent to saying that  $\epsilon = \text{sgn}(X - m(X))$  almost surely, except in case the event  $X - m(X) = 0$  has positive probability.

**Lemma A.3.** *If  $\epsilon$  is medially coupled to  $X$  then*

$$\mathbb{E}[X \mid \epsilon = 1] - \mathbb{E}[X \mid \epsilon = -1] = 2\text{MDM}(X). \quad (16)$$

*Proof.* To prove (16), it suffices to observe that

$$\mathbb{E}[X \mid \epsilon = 1] - \mathbb{E}[X \mid \epsilon = -1] = 2 \cdot \mathbb{E}[\epsilon X] = 2\mathbb{E}|X - m(X)|. \quad (17)$$

$\square$

The proof of the following lemma uses the technique introduced by Erdős in his proof of Theorem 4.1.

**Lemma A.4.** *Suppose  $X_1, \dots, X_n$  and  $\epsilon_1, \dots, \epsilon_n$  are two  $n$ -tuples of random variables such that for all  $i$ ,  $\text{MDM}(X_i) \geq 1$  and  $\epsilon_i$  is medially coupled to  $X_i$ . Suppose the coupled pairs  $\{(X_i, \epsilon_i)\}_{i=1}^n$  are mutually independent of one another. For every sign vector  $\sigma \in \{\pm 1\}^n$ , let*

$$E(\sigma) = \mathbb{E}[X_1 + \dots + X_n \mid (\epsilon_1, \dots, \epsilon_n) = \sigma].$$

*If  $I$  is any half-open interval of length 2, the number of sign vectors  $\sigma$  such that  $E(\sigma) \in I$  is at most  $\binom{n}{\lfloor n/2 \rfloor}$ .*

*Proof.* For any  $i$  we have

$$\mathbb{E}[X_i \mid (\epsilon_1, \dots, \epsilon_n) = \sigma] = \mathbb{E}[X_i \mid \epsilon_i = \sigma_i],$$

since  $\epsilon_j$  is independent of  $\epsilon_i$  for  $j \neq i$ . Thus,

$$E(\sigma) = \sum_{i=1}^n \mathbb{E}[X_i \mid \epsilon_i = \sigma_i].$$

For any two sign vectors  $\sigma, \sigma'$ , we have

$$\begin{aligned} E(\sigma) - E(\sigma') &= \sum_{i=1}^n (\mathbb{E}[X_i \mid \epsilon_i = \sigma_i] - \mathbb{E}[X_i \mid \epsilon_i = \sigma'_i]) \\ &= \sum_{i=1}^n (\sigma_i - \sigma'_i) \text{MDM}(X_i), \end{aligned}$$

where the last line follows from (16). Thus, if  $\sigma \succ \sigma'$  (meaning,  $\sigma_i \geq \sigma'_i$  for all  $i$ , and the inequality is strict for at least one  $i$ ) it follows that  $E(\sigma) - E(\sigma') \geq 2$ . The lemma now follows from Sperner's Lemma (Sperner, 1928), which states that any collection of more than  $\binom{n}{\lfloor n/2 \rfloor}$  sign vectors contains a pair such that  $\sigma \succ \sigma'$ .  $\square$

Recall the statement of Proposition 4.4.

**Proposition A.5.** *If  $X_1, \dots, X_n$  are independent random variables and  $\text{MDM}(X_i) \geq 1$  for all  $i$ , then  $\text{MDM}(X_1 + \dots + X_n) \geq \frac{1}{12}\sqrt{n}$ .*

*Proof.* Let  $\epsilon_1, \dots, \epsilon_n$  be independent random variables medially coupled to  $X_1, \dots, X_n$ . Let  $X = X_1 + \dots + X_n$  and  $m = m(X)$ . Jensen's convex function inequality applied to the random variable  $|X - m|$  implies

$$\mathbb{E}|X - m| = \mathbb{E}_\sigma [\mathbb{E}(|X - m| \mid \sigma)] \geq \mathbb{E}_\sigma [|E(\sigma) - m|].$$

We will prove a lower bound on the quantity appearing on the right-hand side. Let  $k = \lceil \frac{\sqrt{n}}{3} \rceil - 1$ . and apply Lemma A.4 to the intervals  $I_j = (m + 2j - 1, m + 2j + 1]$  for  $-k \leq j \leq k$ . Each of these intervals contains at most  $\binom{n}{\lfloor n/2 \rfloor}$  of the numbers  $E(\sigma)$ . Hence, there are at most  $(2k + 1)\binom{n}{\lfloor n/2 \rfloor}$  sign vectors  $\sigma$  such that  $m - 2k - 1 < E(\sigma) \leq m + 2k + 1$ . The inequality

$$(2k + 1) \binom{n}{\lfloor n/2 \rfloor} = (2 \lceil \frac{\sqrt{n}}{3} \rceil - 1) \binom{n}{\lfloor n/2 \rfloor} < \frac{3}{4} 2^n$$

can be verified by exhaustive enumeration over small values of  $n$  combined with the asymptotic estimate  $\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi n}} \cdot 2^n$  as  $n \rightarrow \infty$ . Therefore, we have

$$\mathbb{E}_\sigma [|E(\sigma) - m|] \geq (2k + 1) \Pr_\sigma (|E(\sigma) - m| \geq 2k + 1) \geq \frac{1}{4} \left( 2 \lceil \frac{\sqrt{n}}{3} \rceil - 1 \right) \geq \frac{1}{4} \lceil \frac{\sqrt{n}}{3} \rceil \geq \frac{1}{12} \sqrt{n}.$$

$\square$